Σ (n = 1 to ∞) 1/(3n² − n) = ?

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Calculate

$$\sum_{n=1}^{\infty} \frac{1}{3n^2 - n} = \frac{3}{2}\ln 3 - \frac{\sqrt{3}\,\pi}{6}$$

Solution by Arkady Alt , San Jose, California, USA.

Since $-\dfrac{\ln(1-x^3)}{x^2} = \dfrac{1}{x^2}\sum_{n=1}^{\infty} \dfrac{x^{3n}}{n}, x \in (0,1)$ then $f(x) := \sum_{n=1}^{\infty} \dfrac{x^{3n-1}}{n(3n-1)} =$

$$\int_0^x -\frac{1}{t^2}\ln(1-t^3)dt = \left(\frac{\ln(1-t^3)}{t} - \ln(1-t) + \frac{\ln(t^2+t+1)}{2} - \sqrt{3}\arctan\left(\frac{2t+1}{\sqrt{3}}\right)\right)_0^x =$$

$$\frac{\ln(1-x^3)}{x} - \ln(1-x) + \frac{\ln(x^2+x+1)}{2} - \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{\sqrt{3}\,\pi}{6}$$

(because $\lim_{t\to 1} \dfrac{\ln(1-t^3)}{t} = \lim_{t\to 1}\left(\dfrac{\ln(1-t^3)}{t^3}\cdot t^2\right) = -1\cdot 0 = 0$).

Noting that $\lim_{x\to 1}\left(\dfrac{\ln(1-x^3)}{x} - \ln(1-x)\right) =$

$$\lim_{x\to 1}\left(\frac{\ln(x^2+x+1)}{x} + \frac{\ln(1-x)}{x} - \ln(1-x)\right) =$$

$\ln 3 + \lim_{x\to 1}\left(\ln(1-x)\cdot\dfrac{1-x}{x}\right) = \ln 3 + \lim_{x\to 1}((1-x)\cdot\ln(1-x)) = \ln 3$ we obtain

$$\sum_{n=1}^{\infty}\frac{1}{3n^2-n} = \lim_{x\to 1} f(x) =$$

$$\lim_{x\to 1}\left(\frac{\ln(1-x^3)}{x} - \ln(1-x) + \frac{\ln(x^2+x+1)}{2} - \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{\sqrt{3}\,\pi}{6}\right) =$$

$$\lim_{x\to 1}\left(\frac{\ln(1-x^3)}{x} - \ln(1-x)\right) + \lim_{x\to 1}\frac{\ln(x^2+x+1)}{2} - \sqrt{3}\lim_{x\to 1}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{\sqrt{3}\,\pi}{6} =$$

$$\ln 3 + \frac{\ln 3}{2} - \sqrt{3}\cdot\frac{\pi}{3} + \frac{\sqrt{3}\,\pi}{6} = \frac{3}{2}\ln 3 - \frac{1}{6}\sqrt{3}\,\pi.$$